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Let ABC be an acute-angled triangle and let H be its orthocenter. Let $h_{\max}$ denote the largest altitude of the triangle ABC. Prove that

$$AH+BH+CH \le 2h_{\max}.$$

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Let R is circumradius of $\triangle ABC$ and let $h_{\max}=h_a$. Since $h_a=\dfrac{bc}{2R}, h_b=\dfrac{ca}{2R}$, $h_c=\dfrac{ab}{2R}$ then $h_a=\max\{h_a,h_b,h_c\} \iff a=\min\{a,b,c\}$.

Noting that $AH=2R\cos A, BH=2R\cos B, CH=2R\cos C$ and $h_a=\dfrac{bc}{2R}=2R\sin B\sin C=R(\cos A+\cos(B-C))$ we obtain

$AH+BH+CH\le 2h_{\max} \iff 2R(\cos A+\cos B+\cos C)\le 2R(\cos A+\cos(C-B)) \iff$

$\cos B+\cos C\le\cos(C-B) \iff 2\cos\dfrac{B+C}{2}\cos\dfrac{C-B}{2}\le 2\cos^2\dfrac{C-B}{2}-1 \iff$

(1) $2\sin\dfrac{A}{2}\cos\dfrac{C-B}{2}\le 2\cos^2\dfrac{C-B}{2}-1.$

WLOG we can assume that $B\le C$.

Then $\begin{cases} 0<A\le B\le C\le\dfrac{\pi}{2} \\ A+B+C=\pi \end{cases} \iff \begin{cases} 0<A\le B \\ B\le\pi-A-B\le\dfrac{\pi}{2} \\ C=\pi-A-B \end{cases} \iff$

$\begin{cases} 0<A\le B \\ \dfrac{\pi}{2}-A\le B\le\dfrac{\pi-A}{2} \\ C=\pi-A-B \end{cases} \iff \begin{cases} 0<A \\ \max\left\{A,\dfrac{\pi}{2}-A\right\}\le B\le\dfrac{\pi-A}{2} \\ A\le\dfrac{\pi-A}{2} \\ C=\pi-A-B \end{cases}$

$\begin{cases} 0<A\le\dfrac{\pi}{3} \\ \max\left\{A,\dfrac{\pi}{2}-A\right\}\le B\le\dfrac{\pi-A}{2} \\ C=\pi-A-B \end{cases}.$

Since $\cos\dfrac{C-B}{2}=\cos\dfrac{\pi-A-2B}{2}=\sin\dfrac{A+2B}{2}$ then

$2\cos^2\dfrac{C-B}{2}-1-2\sin\dfrac{A}{2}\cos\dfrac{C-B}{2}=f(x):=2x^2-2\sin\dfrac{A}{2}\cdot x-1,$

where $x:=\sin\dfrac{A+2B}{2}$. Note that $f(x)$ increase for $x>\dfrac{1}{2}\sin\dfrac{A}{2}$

and $\dfrac{A+\max\{2A,\pi-2A\}}{2}\le\dfrac{A+2B}{2}\le\dfrac{\pi}{2} \iff \dfrac{\max\{3A,\pi-A\}}{2}\le\dfrac{A+2B}{2}\le\dfrac{\pi}{2}.$

Since $\sin\varphi\le\sin\dfrac{A+2B}{2}\le 1$, where $\varphi:=\dfrac{\max\{3A,\pi-A\}}{2}$ and $\sin\varphi>\dfrac{1}{2}\sin\dfrac{A}{2}$

then $f(x)\ge f(\sin\varphi)=\begin{cases} f\left(\sin\dfrac{3A}{2}\right) & \text{if } \dfrac{\pi}{4}\le A\le\dfrac{\pi}{3} \\ f\left(\sin\dfrac{\pi-A}{2}\right) & \text{if } 0<A\le\dfrac{\pi}{4} \end{cases}.$

Let $\dfrac{\pi}{4}\le A\le\dfrac{\pi}{3}$. Then $f\left(\sin\dfrac{3A}{2}\right)=2\sin^2\dfrac{3A}{2}-2\sin\dfrac{A}{2}\cdot\sin\dfrac{3A}{2}-1=$

$-\cos 3A - \cos A + \cos 2A = \cos 2A - 2\cos 2A \cos A = (-\cos 2A)(2\cos A - 1) \geq 0$
because $-\cos 2A \geq 0$ and $2\cos A \geq 1$ for $A \in [\pi/4, \pi/3]$;
Let $0 < A \leq \frac{\pi}{4}$. Then $f\left(\sin \frac{\pi - A}{2}\right) = f\left(\cos \frac{A}{2}\right) = 2\cos^2 \frac{A}{2} - 2\sin \frac{A}{2} \cdot \cos \frac{A}{2} - 1 =$
$\cos A - \sin A \geq 0.$